ON A CLASS OF FUNCTIONAL
EQUATIONS OF MODULAR TYPE

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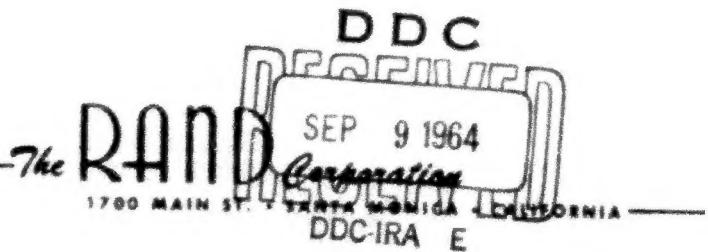
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SUMMARY

The purpose of this paper is to show how the Voronoi functions and their analogues in algebraic number fields, matric fields, and finite fields, can be used to generate large classes of functions possessing functional equations of modular type.

ON A CLASS OF FUNCTIONAL EQUATIONS OF THE MODULAR TYPE

1. INTRODUCTION

The Voronoi functions, or generalized Bessel functions,

$$V_a(x,y) = \int_0^\infty e^{-\pi x^2 s - \pi y^2/s} \frac{ds}{s^a} ,$$

(1)

play an important role in analytic number theory in connection
with lattice point problems and other problems involving zeta-
functions. First introduced by Voronoi in his classic paper
on the Dirichlet divisor problem, they have been discussed by
Hardy [3] and Steen [5], who considered the more general functions
defined by

$$\int_0^\infty V(x;a_1,a_2,\ldots,a_K)x^{s-1}dx = \Gamma(s+a_1)\Gamma(s+a_2) \cdots \Gamma(s+a_K) ,$$

(2)

a class of functions whose importance was recognized by Voronoi.
Generalizations in the direction of algebraic number fields and
matric fields have been discussed by Bellman [1] and Bochner [2].
This function was also used by Hecke in his researches; cf., in
addition, Maass.

In a previous paper [1], we sketched some extensions of the
method used by Hardy [3] to obtain a number of striking identi-
ties satisfied by series formed with these functions. In this
paper, we wish to indicate how to form extensive classes of
functions satisfying functional equations of modular type,
using these functions and their natural extensions in the

rational field, algebraic number fields, matric fields, hyper-complex fields, and finite fields.

A more detailed account will be presented subsequently.

2. THE RATIONAL FIELD

Consider the function defined by the series

$$f(x,y;u,v;t_1,t_2) = \sum_{m,n=-\infty}^{\infty} V_a\left((x+m)\sqrt{t_1}, (y+m)\sqrt{t_2}\right)e^{2\pi i(mu+nv)}, \qquad (1)$$

where $0 < x, y, u, v < 1$, and $t_1, t_2 > 0$. Using the representation in (1.1), we may write

$$f(x,y;u,v;t_1,t_2) = \int_0^\infty \left[\sum_{m=-\infty}^{\infty} e^{-\pi(x+m)^2 s t_1 + 2\pi i mu} \right] \left[\sum_{n=-\infty}^{\infty} e^{-\pi(y+n)^2 t_2/s + 2\pi i nv} \right] \frac{ds}{s} . \qquad (2)$$

Applying the functional equation for the theta-function in each of the brackets, we have

$$f(x,y;u,v;t_1,t_2) = \frac{e^{-2\pi i(xu+yv)}}{\sqrt{t_1 t_2}} f(v,u;-y,-x;t_2^{-1},t_1^{-1}) . \qquad (3)$$

3. AN OBSERVATION

Let us note that no properties of the weight ds/s^a have been used. Consequently, the same functional equation holds for any weight-function $dG(s)$ permitting the interchanges of summation and integration above. Furthermore, contour integrals of the form $\int_c e^{-\pi x^2 z - \pi y^2/z} \frac{dz}{z^a}$ can be used, with different contours yielding different functions.

4. SPECIALIZATION

If we allow x, y, u, and v to assume rational values, a number of interesting identities involving sums over divisors in specified residue classes are obtained.

This is a consequence of the relation

$$V_a(x,y) = x^{2a-2} \int_0^\infty e^{-\pi s - \pi(xy)^2/s} \frac{ds}{s^a} = x^{2a-2} V_a(|xy|), \quad x \neq 0 . \tag{1}$$

Consider, in particular, the case $a = 1$, and $x = y = u = v = 1/2$. Then (2.3) yields, with $t_1 t_2 = t^2$,

$$\sum_{m,n=-\infty}^{\infty} v_a\left(\frac{t|(1+2m)(1+2n)|}{4} \right)(-1)^{m+n} \tag{2}$$

$$= -\frac{1}{t} \sum_{m,n=-\infty}^{\infty} v_a\left(\frac{|(1+2m)(1+2n)|}{4t} \right)(-1)^{m+n} ,$$

or

$$\sum_{R=1}^{\infty} V_a \left(\frac{Rt}{4} \right) a(R) = -\frac{1}{t} \sum_{R=1}^{\infty} V_a \left(\frac{R}{4t} \right) a(R) , \qquad (3)$$

where

$$a(R) = \sum_{|1+2m||1+2n|=R} (-1)^{m+n} , \qquad -\infty < m, n < \infty .$$

Similar identities are obtained with other rational values of x, y, u, and v.

5. GENERALIZATION IN THE RATIONAL FIELD—I

Using the generalized Voronoi function in the form

$$V(x_1, x_2, \ldots, x_{K+1}; a_1, a_2, \ldots, a_K)$$

$$= \int_0^{\infty} \cdots \int_0^{\infty} e^{-s_1 x_1^2 - \cdots - s_K x_K^2 - x_{K+1}^2 / s_1 s_2 \cdots s_K} \frac{ds_1}{s_1^{a_1}} \cdots \frac{ds_K}{s_K^{a_K}} , \qquad (1)$$

cf. Steen [5] and Bellman [1], a relation analogous to that of (2.3) can be obtained for the function defined by the series

$$\sum_{m_1 = -\infty}^{\infty} V((x_1 + m_1)\sqrt{t_1}, \ldots, (x_{K+1} + m_{K+1})\sqrt{t_{K+1}}; a_1, a_2, \ldots, a_k) e^{2\pi i \sum_{i=1}^{K+1} m_i u_i} , \qquad (2)$$

where $0 < x_1, u_1 < 1, t_1 > 0$. These identities yield analogues of (3.3) for higher order divisor functions.

6.   GENERALIZATIONS IN THE RATIONAL FIELD—II

Two further lines of generalization are immediate. In place of the two-dimensional form of (5.1), we may employ

$$\int_0^\infty \int_0^\infty e^{-\pi(s_1 x_1^2 + s_2 x_2^2 + s_1 s_2 x_3^2 - x_4^2/s_1 s_2)} \frac{ds_1}{s_1^{\lambda_1}} \frac{ds_2}{s_2^{\lambda_2}} , \tag{1}$$

utilizing all combinations of products, one at a time, two at a time, and so on. In place of forming sums such as that in (5.2), we can form sums such as

$$\sum_{m_1 = -\infty}^{\infty} V_a\Big(\big[Q((x_1+m_1),(x_2+m_2))\big]^{1/2}, (y+n)\sqrt{t_4}\Big) e^{2\pi i(m_1 u_1 + m_2 u_2 + nu_3)},$$

where $Q(u, v) = t_1 u^2 + 2t_2 uv + t_3 v^2$ is a positive-definite quadratic form. Use of the multidimensional theta-function transformation will yield the desired functional equation.

7.   ALGEBRAIC NUMBER FIELDS

Proceeding in the fashion made classical by the work of Hecke, it is easy to form the corresponding Voronoi functions in totally real fields. Thus, for example, in $R(\sqrt{2})$, we have, as the analogue of the function in (1.1),

$$V_a(x,y) = \int_{s \gg 0} e^{-tr(\pi x^2 s + \pi y^2/s)} \frac{ds_1 ds_2}{\left(s_1^2 - 2s_2^2\right)^\lambda} , \tag{1}$$

where the integration is over the region $s_1 \geq \pm s_2 \sqrt{2}$, and
$x = x_1 + x_2 \sqrt{2}$, $y = y_1 + y_2 \sqrt{2}$.

8. MATRIC FIELDS

Similarly, over matric fields, following the work of Siegel, we may define the function of the symmetric matrices X and Y,

$$V_a(X,Y) = \int_{S > 0} e^{-tr(\pi X^2 S + \pi Y^2 S^{-1})} \frac{dS}{|S|^a} ,$$

where the integration is over the region where the symmetric matrix S is positive-definite, $dS = \prod_{1 \leq i \leq j \leq n} ds_{ij}$, and $|S|$ is the determinant of S, cf. Bellman [1] and Bochner [2].

9. DISCUSSION

In a like fashion, generalized Bessel functions can be defined over many hypercomplex fields for which theta-functions exist. We shall discuss these functions in a subsequent paper.

10. FINITE FIELDS—KLOOSTERMANN SUMS

An analogue of the Voronoi function in finite fields is the Kloostermann sum

$$K(x,q;p) = \sum_{n=1}^{p-1} e^{-2\pi i(x^2 n + n^{-1})q/p} , \tag{1}$$

with many other analogues obtainable from the continuous forms cited above. Corresponding results, which will be presented elsewhere, can be obtained for these functions.

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